LENGTH INEQUALITIES FOR SYSTEMS OF GEODESIC LOOPS ON A SURFACE OF GENUS TWO: 1

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ABSTRACT

We give some length inequality results on systems of simple closed non-dividing geodesics on a compact surface of genus two. One result gives a new characterisation of the octahedral surface, that is, the genus two surface whose symmetry group is a $\mathbf{Z}_2$-extension of that of the platonic solid. This work has applications towards studying Maskit's fundamental domain for the mapping class group in genus two.

Introduction

In this paper we study the moduli space of hyperbolic structures on compact genus two surfaces via systems of simple closed non-dividing geodesics. Let $(\mathscr{S}, \Upsilon)$ be a pair comprising a compact genus two surface endowed with a hyperbolic structure $\mathscr{S}$ together with a prescribed system of simple closed non-dividing geodesics Υ. We show that if Υ satisfies certain length inequalities, then $\mathscr{S}$ is uniquely determined: it is the octahedral surface, that is, its symmetry group is a $\mathbf{Z}_2$-extension of that of the platonic solid (Theorem 1.2).

To be more specific, we shall prescribe six simple closed non-dividing geodesics Υ on $\mathscr{S}$: $\kappa_{0,4}, \kappa_{0,5}, \lambda_2, \lambda_0, \kappa_{3,0}, \kappa_{1,2}$. We require that $\kappa_{0,4}, \kappa_{0,5}$ are shorter than any other simple closed non-dividing geodesic loop on $\mathscr{S}$, and that $l[\lambda_2] \leqslant l[\lambda_0]$ and $l[\kappa_{3,0}] \leqslant l[\kappa_{1,2}]$. If all these length inequalities are satisfied, then $\mathscr{S}$ is the octahedral surface.

In order to prove Theorem 1.2, we first need to prove Theorem 1.1. Again we prescribe six simple closed non-dividing geodesics Υ': $\kappa_{0,4}, \kappa_{3,4}, \kappa_{0,5}, \kappa_{3,5}, \lambda_2, \lambda_0$. If $l[\kappa_{0,4}] \leqslant l[\kappa_{3,4}]$, $l[\kappa_{0,5}] \leqslant l[\kappa_{3,5}]$, $l[\lambda_2] \leqslant l[\lambda_0]$, then we show that each of these inequalities must be an equality. It will follow that a subsurface of $\mathscr{S}$ (spanned by Υ') exhibits an orientation-reversing involution Φ.

This work has applications to the author's study **[4]** of Maskit's fundamental domain for the mapping class group **[10]** in the special case of genus two. Also, the work is closely related to that of Schmutz for systems of shortest simple closed geodesics—what he refers to as *systoles* **[12]**.

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1. *Preliminaries and results*

It is well known that every compact hyperbolic surface $\mathscr{S}$ of genus two admits an involution $\mathscr{J}$: the hyperelliptic involution. The involution $\mathscr{J}$ has six fixed points, the Weierstrass points, and the quotient orbifold $\mathscr{O} \cong \mathscr{S}/\mathscr{J}$ is a sphere with six order two

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cone points. Every simple closed geodesic on $\mathcal{S}$ is setwise fixed by $\mathcal{J}$ (see Haas and Susskind [5]). It is a consequence that every simple closed non-dividing geodesic passes through precisely two Weierstrass points. Equivalently, every simple closed non-dividing geodesic on $\mathcal{S}$ projects to a simple geodesic between distinct cone points on $\mathcal{O}$.

For simplicity of exposition, we shall work on the orbifold quotient $\mathcal{O}$, but all constructions can be performed equivariantly with respect to $\mathcal{J}$ on $\mathcal{S}$. So the results will be expressed in terms of length inequalities on systems of arcs on $\mathcal{O}$, rather than on systems of non-dividing geodesics on $\mathcal{S}$.

DEFINITIONS. A simple geodesic between distinct cone points on $\mathcal{O}$ is an *arc*. Two distinct arcs *cross* if they intersect in a point that is not a cone point.

We adopt an arc-labelling scheme compatible with [4]: label the cone points on $\mathcal{O}$ by $c_0, \ldots, c_5$; let

$$K := \kappa_{0,1} \cup \kappa_{1,2} \cup \kappa_{2,3} \cup \kappa_{3,0} \cup \kappa_{0,4} \cup \kappa_{1,4} \cup \kappa_{2,4} \cup \kappa_{3,4} \cup \kappa_{0,5} \cup \kappa_{1,5} \cup \kappa_{2,5} \cup \kappa_{3,5},$$

where $\kappa_{i,j}$ is an arc between c_i, c_j and $\kappa_{i,j}, \kappa_{k,l}$ do not cross; and let $\Lambda := \lambda_0 \cup \lambda_2$, where λ_0 (respectively λ_2) is an arc between c_4, c_5 crossing only $\kappa_{0,1} \subset K$ (respectively $\kappa_{2,3} \subset K$). Note that K has the combinatorial edge pattern of an octahedron.

We shall abuse notation slightly: we use *Saccheri* to refer to a finite-area hyperbolic quadrilateral with two adjacent right-angles (compare [2, p. 156]).

THEOREM 1.1. *We have the following: if* $l[\kappa_{0,4}] \leqslant l[\kappa_{3,4}]$, $l[\kappa_{0,5}] \leqslant l[\kappa_{3,5}]$, $l[\lambda_2] \leqslant l[\lambda_0]$, *then* $l[\kappa_{0,4}] = l[\kappa_{3,4}]$, $l[\kappa_{0,5}] = l[\kappa_{3,5}]$, $l[\lambda_2] = l[\lambda_0]$.

In [4] we construct an orbifold $\mathcal{E}$ with $l[\kappa_{0,4}] = l[\kappa_{3,4}] = l[\kappa_{0,5}] = l[\kappa_{3,5}] = l[\lambda_2] = l[\lambda_0] = l[\kappa_{1,4}] = l[\kappa_{2,4}] = l[\kappa_{1,5}] = l[\kappa_{2,5}]$ and $l[\kappa_{3,0}] = l[\kappa_{1,2}]$. The orbifold $\mathcal{E}$ has conformal symmetry group $\mathbf{Z}_2 \times \mathbf{Z}_2$, and $\mathcal{E}$—or rather its double cover—occurs in a conjecture we make in [4] for the Maskit domain in genus two.

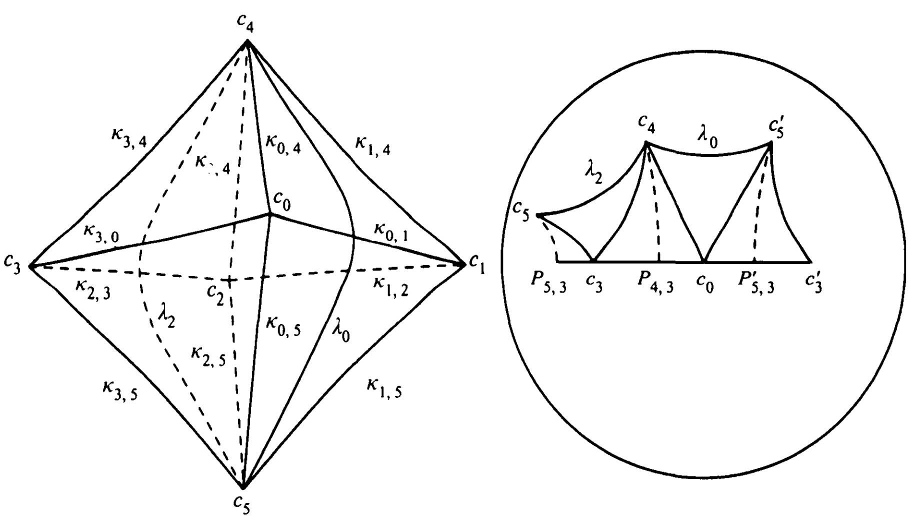

FIG. 1. The arc set $K \cup \Lambda$ and lift of Ω

Proof. The arc set Λ divides $\mathcal{O}$ into two components. Let $\mathcal{O}_3$ denote the component of $\mathcal{O} \backslash \Lambda$ containing $\kappa_{3,0}$.

Cut $\mathcal{O}_3$ open along $\kappa_{3,0}$ so as to obtain an annulus. We then make a further cut along $\kappa_{3,5}$ so as to obtain a simply connected domain Ω. Choose a lift of Ω in the universal cover of the annulus $\mathcal{O}_3\tilde{\setminus}\kappa_{3,0} \subset \mathbf{H}^2$. Label the geodesics around the boundary of Ω by $\kappa_{3,0}, \kappa_{3,5}, \lambda_2, \lambda_0, \kappa'_{3,5}$, in cyclic order. Without confusion, give the lifts of $\kappa_{0,4}$, $\kappa_{0,5}, \kappa_{3,4}$ having non-trivial intersection with Ω the same labels. In the same cyclic order, label orbits of cone points around the boundary of Ω: $c_0, c_3, c_5, c_4, c'_5, c'_3$, so that $*_i \in \tilde{c}_i$.

We suppose that $l[\kappa_{0,4}] \leqslant l[\kappa_{3,4}]$ and $l[\kappa_{0,5}] \leqslant l[\kappa_{3,5}]$, and show that $l[\lambda_2] \geqslant l[\lambda_0]$. It should be clear from the method that if either of the first two inequalities is strict, then the third inequality must also be strict.

Let $P_{5,3}, P_{4,3}, P'_{5,3}$ denote the perpendiculars to $\kappa_{3,0}$ from c_5, c_4, c'_5, respectively.

By supposition, $d[c_4, c_0] = l[\kappa_{0,4}] \leqslant l[\kappa_{3,4}] = d[c_4, c_3]$, and so $P_{4,3}$ must be closer to c_0 than to c_3. Similarly, $d[c'_5, c_0] = l[\kappa_{0,5}] \leqslant l[\kappa_{3,5}] = d[c'_5, c'_3]$, and so $P'_{5,3}$ must be closer to c_0 than to c'_3.

Now $d[c_3, c_0] = d[c_0, c'_3] = l[\kappa_{3,0}]$, so we have shown that $d[P_{4,3}, P'_{5,3}] \leqslant l[\kappa_{3,0}]$. Moreover, $d[P_{5,3}, P'_{5,3}] = 2l[\kappa_{3,0}]$, and so $d[P_{5,3}, P_{4,3}] \geqslant l[\kappa_{3,0}]$.

Label the quadrilaterals bounded by $\lambda_2 \cup P_{5,3} \cup \kappa_{3,0} \cup P_{4,3}$ and $\lambda_0 \cup P_{4,3} \cup \kappa_{3,0} \cup P'_{5,3}$ by $\mathcal{Q}_{2,3}$ and $\mathcal{Q}_{0,3}$, respectively. Both $\mathcal{Q}_{2,3}$ and $\mathcal{Q}_{0,3}$ are Saccheri and so, since the $\kappa_{3,0}$-edge of $\mathcal{Q}_{2,3}$ (between $P_{5,3}, P_{4,3}$) is longer than the $\kappa_{3,0}$-edge of $\mathcal{Q}_{0,3}$ (between $P_{4,3}, P'_{5,3}$), the λ_2-edge of $\mathcal{Q}_{2,3}$ must be longer than the λ_0-edge of $\mathcal{Q}_{0,3}$.

At this point we note that if $\mathcal{O}$ satisfies the hypotheses of Theorem 1.1, then $\mathcal{O}_3$ has an orientation-reversing involution Φ. The involution Φ is such that $\Phi(\kappa_{0,4}) = \kappa_{3,4}$, $\Phi(\kappa_{0,5}) = \kappa_{3,5}$ and $\Phi(\lambda_2) = \lambda_0$.

THEOREM 1.2. *If both $\kappa_{0,4}$ and $\kappa_{0,5}$ are shortest arcs, and both $l[\lambda_2] \leqslant l[\lambda_0]$ and $l[\kappa_{3,0}] \leqslant l[\kappa_{1,2}]$, then $\mathcal{O}$ is the octahedral orbifold.*

Proof. Recall that $\mathcal{O}_3$ denotes the component of $\mathcal{O}\setminus\Lambda$ containing $\kappa_{3,0}$, and let $\mathcal{O}_1$ denote the component of $\mathcal{O}\setminus\Lambda$ containing $\kappa_{1,2}$.

By hypothesis, $\kappa_{0,4}, \kappa_{0,5}$ are shortest arcs, and so $l[\kappa_{0,4}] \leqslant l[\kappa_{3,4}]$, $l[\kappa_{0,5}] \leqslant l[\kappa_{3,5}]$. Also by hypothesis, $l[\lambda_2] \leqslant l[\lambda_0]$, so Theorem 1.1 implies that $l[\kappa_{0,4}] = l[\kappa_{3,4}]$, $l[\kappa_{0,5}] = l[\kappa_{3,5}]$ and $l[\lambda_2] = l[\lambda_0]$.

It follows that $\mathcal{O}_3$ has full symmetry group $\mathbf{Z}_2 \times \mathbf{Z}_2$ generated by a pair of orientation-reversing involutions Φ, Ψ. The involution Φ is as noted above; the involution Ψ is such that $\Psi(\kappa_{0,4}) = \kappa_{0,5}$, $\Psi(\kappa_{3,4}) = \kappa_{3,5}$, $\Psi(\lambda_0) = \lambda_0$ and $\Psi(\lambda_2) = \lambda_2$.

Let $\angle c_i \mathcal{O}_j$ denote the angle between λ_2, λ_0 in $\mathcal{O}_j$ at c_i, for $i = 4, 5$, $j = 1, 3$. The Ψ involution of $\mathcal{O}_3$ implies that $\angle c_4 \mathcal{O}_3 = \angle c_5 \mathcal{O}_3$. We use this angle condition together with $l[\lambda_2] = l[\lambda_0]$ to show that $\mathcal{O}_1$ also has rotational symmetry. In conjunction with $l[\kappa_{3,0}] \leqslant l[\kappa_{1,2}]$, we show that $\mathcal{O}_1$ is isometric to $\mathcal{O}_3$.

Consider $\mathcal{O}_1$. Let $P_{4,2}, P_{5,1}$ denote the perpendiculars to $\kappa_{1,2}$ from c_4, c_5, respectively. Cut $\mathcal{O}_1$ open along $\kappa_{1,2}$ so as to obtain an annulus. The perpendiculars $P_{4,1}, P_{5,1}$ divide $\mathcal{O}_1\setminus\kappa_{1,2}$ into two Saccheri quadrilaterals $\mathcal{Q}_{2,1}, \mathcal{Q}_{0,1}$, where $\mathcal{Q}_{2,1}$ has λ_2 on its boundary and $\mathcal{Q}_{0,1}$ has λ_0 on its boundary. Since $l[\lambda_2] = l[\lambda_0]$, it follows (for the same reason as in Theorem 1.1) that the $\kappa_{1,2}$-edges of $\mathcal{Q}_{2,1}$, $\mathcal{Q}_{0,1}$ must be the same length. Therefore $\mathcal{Q}_{2,1}$, $\mathcal{Q}_{0,1}$ must be mirror images. In particular, $\angle c_i \mathcal{Q}_{2,1} = \angle c_i \mathcal{Q}_{0,1}$, $i = 4, 5$.

By the Ψ involution of $\mathcal{O}_3$, $\angle c_4 \mathcal{O}_3 = \angle c_5 \mathcal{O}_3$, and so $\angle c_4 \mathcal{O}_1 = \angle c_5 \mathcal{O}_1$; that is, $\angle c_4 \mathcal{Q}_{2,1} + \angle c_4 \mathcal{Q}_{0,1} = \angle c_5 \mathcal{Q}_{2,1} + \angle c_5 \mathcal{Q}_{0,1}$.

It follows that $\angle c_4 \mathcal{Q}_{2,1} = \angle c_4 \mathcal{Q}_{0,1} = \angle c_5 \mathcal{Q}_{2,1} = \angle c_5 \mathcal{Q}_{0,1}$. Therefore $\mathcal{Q}_{2,1}, \mathcal{Q}_{0,1}$ are isometric by an *orientation-preserving* isometry. So $\mathcal{O}_1 \backslash \kappa_{1,2}$ has rotational symmetry exchanging $\mathcal{Q}_{2,1}, \mathcal{Q}_{0,1}$. Gluing along $\kappa_{1,2}$ so as to recover $\mathcal{O}_1$, this symmetry is respected.

Let $p_{2,1}, p_{0,1}$ denote the common perpendiculars between $\lambda_2, \kappa_{1,2}$ and between λ_0, $\kappa_{1,2}$, respectively, in $\mathcal{O}_1$. We have shown that $\mathcal{Q}_{2,1}, \mathcal{Q}_{0,1}$ are both symmetric. It follows that $p_{2,1}$ divides $\mathcal{Q}_{2,1}$ into equal halves, and $p_{0,1}$ divides $\mathcal{Q}_{0,1}$ into equal halves. So $p_{2,1}$, $p_{0,1}$ divide $\kappa_{1,2}$ (as a boundary component of $\mathcal{O}_1 \backslash \kappa_{1,2}$) into two equal length halves.

Let $d[c_4, p_{2,1} \cap \kappa_{1,2}]$ denote the distance between $c_4, p_{2,1} \cap \kappa_{1,2}$ in $\mathcal{Q}_{2,1}$, and $d[c_4, p_{0,1} \cap \kappa_{1,2}]$ the distance between $c_4, p_{0,1} \cap \kappa_{1,2}$ in $\mathcal{Q}_{0,1}$. Since $\mathcal{Q}_{2,1}, \mathcal{Q}_{0,1}$ are isometric, we have that $d[c_4, p_{2,1} \cap \kappa_{1,2}] = d[c_4, p_{0,1} \cap \kappa_{1,2}]$. One of c_1, c_2 must lie in the half of $\kappa_{1,2}$ nearer to c_4. So $\min\{d[c_4, c_1], d[c_4, c_2]\} \leqslant d[c_4, p_{2,1} \cap \kappa_{1,2}]$.

In $\mathcal{O}_3$, let $p_{2,3}, p_{0,3}$ denote the common perpendiculars between $\lambda_2, \kappa_{3,0}$ and between $\lambda_0, \kappa_{3,0}$, respectively. By $\mathbf{Z}_2 \times \mathbf{Z}_2$ symmetry, we know that c_3 lies at $p_{2,3} \cap \kappa_{3,0}$ and c_0 lies at $p_{0,3} \cap \kappa_{3,0}$. In particular, $d[c_4, p_{2,3} \cap \kappa_{3,0}] = d[c_4, c_3] = l[\kappa_{3,4}]$.

By hypothesis, $l[\kappa_{3,0}] \leqslant l[\kappa_{1,2}]$. So, from elementary geometry, $d[c_4, p_{2,3} \cap \kappa_{3,0}] \geqslant d[c_4, p_{2,1} \cap \kappa_{1,2}]$, with equality *if and only if* $l[\kappa_{3,0}] = l[\kappa_{1,2}]$.

So, *unless* $l[\kappa_{3,0}] = l[\kappa_{1,2}]$ and c_1, c_2 lie at $p_{2,1} \cap \kappa_{1,2}, p_{0,1} \cap \kappa_{1,2}$, the arc $\kappa_{0,4}$ will be strictly longer than an arc between c_4, c_1 or c_4, c_2. This would contradict $\kappa_{0,4}$ being a shortest arc. So c_1, c_2 must lie at $p_{2,1} \cap \kappa_{1,2}, p_{0,1} \cap \kappa_{1,2}$, and $l[\kappa_{3,0}] = l[\kappa_{1,2}]$. Therefore $\mathcal{O}_1$ is isometric to $\mathcal{O}_3$.

Gluing along Λ to recover $\mathcal{O}$, there are four shortest arcs incident at c_4 and at c_5. Four shortest arcs incident at a cone point is a known characterisation of the octahedral orbifold (see, for example, [**12**, p. 589, Lemma 5.1]). Alternatively, it is an exercise to extend the shortest arc set to that of the octahedral orbifold.

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